Some bounds on convex combinations of ω and χ for decompositions into many parts

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Abstract

A *k–decomposition* of the complete graph K_n is a decomposition of K_n into k spanning subgraphs $G_1, \dots, G_k$. For a graph parameter p, let $p(k; K_n)$ denote the maximum of $\sum_{j=1}^{k} p(G_j)$ over all k–decompositions of K_n. It is known that $\chi(k; K_n) = \omega(k; K_n)$ for $k \leq 3$ and conjectured that this equality holds for all k. In an attempt to get a handle on this, we study convex combinations of ω and χ; namely, the graph parameters $A_r(G) = (1-r)\omega(G) + r\chi(G)$ for $0 \leq r \leq 1$. It is proven that $A_r(k; K_n) \leq n + \binom{k}{2}$ for small r. In addition, we prove some generalizations of a theorem of Kostochka, et al. [1].

1 Introduction

A *k–decomposition* of the complete graph K_n is a decomposition of K_n into k spanning subgraphs $G_1, \dots, G_k$; that is, the G_j have the same vertices as K_n and each edge of K_n belongs to precisely one of the G_j. For a graph parameter p and a positive integer k, define

$$p(k; K_n) = \max\{\sum_{j=1}^{k} p(G_j) \mid (G_1, \dots, G_k) \text{ a } k\text{–decomposition of } K_n\}.$$

We say $(G_1, \dots, G_k)$ is a *p-optimal* k-decomposition of K_n if $\sum_{j=1}^{k} p(G_j) = p(k; K_n)$. We will be interested in parameters that are convex combinations of the clique number and the chromatic number of a graph G. For $0 \leq r \leq 1$, define $A_r(G) = (1-r)\omega(G) + r\chi(G)$. We would like to determine $A_r(k; K_n)$. The following theorem of Kostochka, et al. does this for the case $r = 0$.

Theorem 1 (Kostochka, et al. [1]). *If k and n are positive integers, then $\omega(k; K_n) \leq n + \binom{k}{2}$. If $n \geq \binom{k}{2}$, then $\omega(k; K_n) = n + \binom{k}{2}$.*

Since $A_r(k; K_n) \leq (1-r)\omega(k; K_n) + r\chi(k; K_n)$, this theorem combined with the following result of Watkinson gives the general upper bound

$$A_r(k; K_n) \leq n + (1-r)\binom{k}{2} + r\frac{k!}{2}. \tag{1}$$

Theorem 2 (Watkinson [3]). *If k and n are positive integers, then $\chi(k; K_n) \leq n + \frac{k!}{2}$.*

From Theorem 1, we see that $A_r(k; K_n) \leq n + \binom{k}{2}$ is the best possible bound. Equation (1) shows that this holds for $k \leq 3$. Also, this bound is an immediate consequence of a conjecture made by Plesník.

Conjecture 3 (Plesník [2]). *If k and n are positive integers, then $\chi(k; K_n) \leq n + \binom{k}{2}$.*

Since $\omega \leq \chi$, if the conjectured bound on $A_r(k; K_n)$ holds for r, then it holds for all $0 \leq s \leq r$ as well. This suggests that it may be easier to look at small values of r first. Our next theorem proves the optimal bound for small r.

Theorem 11. *Let k and n be positive integers and $0 \leq r \leq \min\{1, 3/k\}$. Then*

$$A_r(k; K_n) \leq n + \binom{k}{2}.$$

Along the way we prove some generalizations of Theorem 1. A definition is useful here. For $0 \leq m \leq k$, define

$$\chi_m(k; K_n) = \max\{\sum_{j=1}^{m} \chi(G_j) + \sum_{j=m+1}^{k} \omega(G_j) \mid (G_1, \ldots, G_k) \text{ a } k\text{–decomposition of } K_n\}.$$

We say $(G_1, \ldots, G_k)$ is a χ_m-*optimal* k-decomposition of K_n if $\sum_{j=1}^{m} \chi(G_j) + \sum_{j=m+1}^{k} \omega(G_j) = \chi_m(k; K_n)$. Note that $\chi_0(k; K_n) = \omega(k; K_n)$ and $\chi_k(k; K_n) = \chi(k; K_n)$.

We prove that the following holds for a given value of m if and only if Conjecture 3 holds for $k = m$.

Conjecture 7. *Let m and $n \geq 1$ be non-negative integers. Then $\chi_m(k; K_n) \leq n + \binom{k}{2}$ for all $k \geq m$.*

In the last section, we prove similar results for decompositions of K_n^r into r-uniform hypergraphs.

2 Notation

We quickly fix some terminology and notation.

A *hypergraph* G is a pair consisting of finite set $V(G)$ together with a set $E(G)$ of subsets of $V(G)$ of size at least two. The elements of $V(G)$ and $E(G)$ are called *vertices* and *edges* respectively. If $|e| = r$ for all $e \in E(G)$, then G is r-*uniform*. A 2-uniform hypergraph is a *graph*. The *order* $|G|$ of G is the number of vertices in G. The *size* $s(G)$ of G is the number of edges in G. The *degree* $d(v)$ of a vertex $v \in V(G)$ is the number of edges of G that contain v. Vertices $v_1, \ldots, v_t$ are called *adjacent* in G if $\{v_1, \ldots, v_t\} \in E(G)$.

Given two hypergraphs G and H, we say that H is a *subhypergraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

Given a hypergraph G and $X \subseteq V(G)$, let $G[X]$ denote the hypergraph with vertex set X and edge set $\{e \in E(G) \mid e \subseteq X\}$. This is called the subhypergraph of G *induced* by X. Let $G - X$ denote $G[V(G) \setminus X]$. For $e \subseteq V(G)$, let $G + e$ and $G - e$ denote the hypergraphs with vertex set $V(G)$

and edge sets $E(G) \cup \{e\}$ and $E(G) \setminus \{e\}$ respectively.
Given an r-uniform hypergraph G, $X \subseteq V(G)$ is a *clique* if $E(G[X])$ contains every r-subset of X. The *clique number* $\omega(G)$ is the maximum size of a clique in G. If $\omega(G) = |G|$, then G is called *complete*. Denote the r-uniform complete hypergraph on n vertices by K_n^r. For the case of graphs($r = 2$) we drop the superscript, writing K_n.
For a graph G, the *chromatic number* $\chi(G)$ of G is the least number of labels required to label the vertices so that adjacent vertices receive distinct labels. Note that if $\{X_1, \dots, X_t\}$ is a partition of $V(G)$, then $\chi(G) \leq \sum_{j=1}^{t} \chi(G[X_j])$. Following [1] we call this property *subadditivity* of χ.

3 Convex combinations of ω and χ

Given a graph G, let $P(G)$ denote the induced subgraph of G on the vertices of positive degree; that is,

$$P(G) = G[\{v \in V(G) \mid d(v) \geq 1\}].$$

Lemma 4. *Let $0 \leq m < k$ and n a positive integer. If $(G_1, \dots, G_k)$ is a χ_m-optimal k-decomposition of K_n with $s(G_1)$ maximal, then $P(G_j)$ is complete for $m < j \leq k$.*

Proof. Let $(G_1, \dots, G_k)$ be a χ_m-optimal k-decomposition of K_n with $s(G_1)$ maximal. Let $m < j \leq k$. Take $e \in E(G_j)$. Then $(G_1 + e, \dots, G_j - e, \dots, G_k)$ is a k-decomposition of K_n with $s(G_1 + e) > s(G_1)$. Hence $(G_1 + e, \dots, G_j - e, \dots, G_k)$ is not χ_m-optimal, which implies that $\omega(G_j - e) < \omega(G_j)$. Whence every edge of G_j is involved in every maximal clique and thus every vertex of positive degree is involved in every maximal clique. Hence $\omega(P(G_j)) = |P(G_j)|$, showing $P(G_j)$ complete. ☐

Theorem 5. *Let $m \geq 1$. Assume $\chi(m; K_n) \leq n + f(m)$ for all $n \geq 1$. Then, for $k \geq m$,*

$$\chi_m(k; K_n) \leq n + \binom{k}{2} + f(m) - \binom{m}{2}.$$

Proof. Fix $k \geq m$. Let $(G_1, \dots, G_k)$ be a χ_m-optimal k-decomposition of K_n with $s(G_1)$ maximal. Set $X = \bigcup_{j=m+1}^{k} V(P(G_j))$. Then $(G_1 - X, \dots, G_m - X)$ is an m-decomposition of $K_{n-|X|}$ and hence

$$\sum_{j=1}^{m} \chi(G_j - X) \leq n - |X| + f(m). \tag{2}$$

Fix $1 \leq j \leq m$. By Lemma 4, $P(G_i)$ is complete for $i > m$. Hence $P(G_j[X])$ and $P(G_i[X])$ have at most one vertex in common for $i > m$. Thus $|P(G_j[X])| \leq k - m$. In particular, $\chi(G_j[X]) = \chi(P(G_j[X])) \leq k - m$. Combining this with (2), we have

$$\sum_{j=1}^{m} \chi(G_j - X) + \sum_{j=1}^{m} \chi(G_j[X]) \leq n - |X| + f(m) + m(k - m).$$

By subadditivity of χ, this is

$$\sum_{j=1}^{m} \chi(G_j) \leq n - |X| + f(m) + m(k-m). \tag{3}$$

Also, since $P(G_i)$ is complete for $i > m$,

$$\sum_{i=m+1}^{k} \omega(G_i) = \sum_{i=m+1}^{k} |P(G_i)| \leq |X| + \binom{k-m}{2}.$$

Adding this to (3) yields

$$\chi_m(k; K_n) = \sum_{j=1}^{m} \chi(G_j) + \sum_{i=m+1}^{k} \omega(G_i) \leq n + \binom{k-m}{2} + f(m) + m(k-m),$$

which is the desired inequality since $\binom{k-m}{2} + m(k-m) = \binom{k}{2} - \binom{m}{2}$. □

Corollary 6. *Let $m \geq 1$. Assume $\chi(m; K_n) \leq n + \binom{m}{2}$ for all $n \geq 1$. Then, for $k \geq m$,*

$$\chi_m(k; K_n) \leq n + \binom{k}{2}.$$

This shows that the following holds for a given value of m if and only if Conjecture 3 holds for $k = m$.

Conjecture 7. *Let m and $n \geq 1$ be non-negative integers. Then $\chi_m(k; K_n) \leq n + \binom{k}{2}$ for all $k \geq m$.*

Since $\chi(1; K_n) \leq n$, we immediately have a generalization of Theorem 1.

Corollary 8. *If k and n are positive integers, then $\chi_1(k; K_n) \leq n + \binom{k}{2}$. If $n \geq \binom{k}{2}$ then $\chi_1(k; K_n) = n + \binom{k}{2}$.*

With the help of Theorem 2, we get a stronger generalization.

Corollary 9. *If $k \geq 3$ and n are positive integers, then $\chi_3(k; K_n) \leq n + \binom{k}{2}$. If $n \geq \binom{k}{2}$ then $\chi_3(k; K_n) = n + \binom{k}{2}$.*

We don't know if Conjecture 7 holds for any larger value of m.

Corollary 10. *Let k and n be positive integers with $n \geq \binom{k}{2}$. If A is a graph appearing in an ω-optimal k-decomposition of K_n, then $\chi(A) = \omega(A)$.*

Proof. Let $(A, G_2, \ldots, G_k)$ be an ω-optimal k-decomposition of K_n. Then, by Theorem 1,

$$\omega(A) + \sum_{j=2}^{k} \omega(G_j) = n + \binom{k}{2}.$$

Hence, by Corollary 8,

$$n+\binom{k}{2}=\omega(A)+\sum_{j=2}^{k}\omega(G_j)\le\chi(A)+\sum_{j=2}^{k}\omega(G_j)\le n+\binom{k}{2}.$$

Thus,

$$\omega(A)+\sum_{j=2}^{k}\omega(G_j)\le\chi(A)+\sum_{j=2}^{k}\omega(G_j),$$

which gives $\chi(A)=\omega(A)$ as desired.

$\square$

Theorem 11. *Let k and n be positive integers and $0\le r\le\min\{1,3/k\}$. Then*

$$A_r(k;K_n)\le n+\binom{k}{2}.$$

Proof. If $k\le 3$, then $r=1$ and the assertion follows from Corollary 9. Assume $k>3$. Let $(G_1,\ldots,G_k)$ be a k–decomposition of K_n. Since any rearrangement of $(G_1,\ldots,G_k)$ is also a k–decomposition of K_n, Corollary 9 gives us the $\binom{k}{3}$ permutations of the inequality

$$\chi(G_1)+\chi(G_2)+\chi(G_3)+\omega(G_4)+\ldots+\omega(G_k)\le n+\binom{k}{2}.$$

Adding these together gives

$$\binom{k-1}{3}\sum_{j=1}^{k}\omega(G_j)+\binom{k-1}{2}\sum_{j=1}^{k}\chi(G_j)\le\binom{k}{3}\left(n+\binom{k}{2}\right),$$

which is

$$\frac{k-3}{k}\sum_{j=1}^{k}\omega(G_j)+\frac{3}{k}\sum_{j=1}^{k}\chi(G_j)\le n+\binom{k}{2}.$$

Combining the sums yields

$$\sum_{j=1}^{k}A_r(G_j)\le\sum_{j=1}^{k}A_{\frac{3}{k}}(G_j)=\sum_{j=1}^{k}\left(\frac{k-3}{k}\omega(G_j)+\frac{3}{k}\chi(G_j)\right)\le n+\binom{k}{2}.$$

$\square$

4 Clique number of uniform hypergraphs

A *k–decomposition* of the complete r-uniform hypergraph K_n^r is a decomposition of K_n^r into k spanning subhypergraphs $G_1,\ldots,G_k$; that is, the G_j have the same vertices as K_n^r and each edge of K_n^r belongs to precisely one of the G_j. Let

$$\omega(k; K_n^r) = \max\{\sum_{j=1}^{k} \omega(G_j) \mid (G_1, \ldots, G_k) \text{ a } k\text{–decomposition of } K_n^r\}.$$

We say $(G_1, \ldots, G_k)$ is a ω-*optimal* k-decomposition of K_n^r if $\sum_{j=1}^{k} \omega(G_j) = \omega(k; K_n^r)$.

Given an r-uniform hypergraph G, let $P(G)$ denote the induced subhypergraph of G on the vertices of positive degree; that is,

$$P(G) = G[\{v \in V(G) \mid d(v) \geq 1\}].$$

Lemma 12. *Let k, n, and $r \geq 2$ be positive integers. If $(G_1, \ldots, G_k)$ is an ω-optimal k-decomposition of K_n^r with $s(G_1)$ maximal, then $P(G_j)$ is complete for $j \geq 2$.*

Proof. Let $(G_1, \ldots, G_k)$ be am ω-optimal k-decomposition of K_n^r with $s(G_1)$ maximal. Let $j \geq 2$. Take $e \in E(G_j)$. Then $(G_1 + e, \ldots, G_j - e, \ldots, G_k)$ is a k-decomposition of K_n^r with $s(G_1 + e) > s(G_1)$. Hence $(G_1 + e, \ldots, G_j - e, \ldots, G_k)$ is not ω-optimal, which implies that $\omega(G_j - e) < \omega(G_j)$. Whence every edge of G_j is involved in every maximal clique and thus every vertex of positive degree is involved in every maximal clique. Hence $\omega(P(G_j)) = |P(G_j)|$, showing $P(G_j)$ complete. □

Theorem 13. *Let k, n, and $r \geq 2$ be positive integers. Then $\omega(k; K_n^r) \leq n + (r-1)\binom{k}{2}$ and if $n \geq (r-1)\binom{k}{2}$, then $\omega(k; K_n^r) = n + (r-1)\binom{k}{2}$.*

Proof. Let $(G_1, \ldots, G_k)$ be a ω-optimal k-decomposition of K_n^r with $s(G_1)$ maximal.
Set $X = \bigcup_{j=2}^{k} V(P(G_j))$. By Lemma 12, $P(G_j)$ is complete for $j \geq 2$. Hence $P(G_j[X])$ and $P(G_1[X])$ have at most $r-1$ vertices in common for $j \geq 2$. Thus $|P(G_1[X])| \leq (r-1)(k-1)$. In particular, $\omega(G_1[X]) = \omega(P(G_1[X])) \leq (r-1)(k-1)$. We have

$$\begin{aligned}
\omega(k; K_n^r) = \sum_{j=1}^{k} \omega(G_j) &\leq \omega(G_1 - X) + \omega(G_1[X]) + \sum_{j=2}^{k} \omega(G_j) \\
&\leq n - |X| + (r-1)(k-1) + \sum_{j=2}^{k} \omega(G_j) \\
&= n - |X| + (r-1)(k-1) + \sum_{j=2}^{k} |P(G_j)| \\
&\leq n - |X| + (r-1)(k-1) + |X| + (r-1)\binom{k-1}{2} \\
&= n + (r-1)\binom{k}{2}.
\end{aligned}$$

This proves the upper bound. To get the lower bound, we generalize a construction in [1]. The construction for $n = (r-1)\binom{k}{2}$ can be extended for each additional vertex by adding all the edges

involving the new vertex to a single hypergraph in the decomposition. Thus, it will be enough to take care of the case $n = (r-1)\binom{k}{2}$.
Let $V(K_n^r) = \{(i,j) \mid 1 \le i < j \le k\} \times \{1,\ldots,r-1\}$. For $1 \le t \le k$, we define a hypergraph G_t. Let $V(G_t)$ be the set of vertices of K_n^r whose names have t in one of the coordinates of the leading ordered pair. Let $E(G_t)$ be all r-subsets of $V(G_t)$. We have $|V(G_t)| = (r-1)(k-1)$. In addition, the G_t are pairwise edge disjoint since $i \ne j \Rightarrow V(G_i) \cap V(G_j) \le r-1$. Whence $(G_1,\ldots,G_k)$ can be extended to a k-decomposition of K_n^r, giving

$$\omega(k;K_n^r) \ge \sum_{j=1}^{k} \omega(G_j) = k(r-1)(k-1) = (r-1)\binom{k}{2} + (r-1)\binom{k}{2} = n + (r-1)\binom{k}{2}.$$

$\square$

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